Degree of polarization for optical near fields

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We investigate an extension to the concept of degree of polarization that applies to arbitrary electromagnetic fields, i.e., fields whose wave fronts are not necessarily planar. The approach makes use of generalized spectral Stokes parameters that appear as coefficients, when the full 3×3 spectral coherence matrix is expanded in terms of the Gell-Mann matrices. By defining the degree of polarization in terms of these parameters in a manner analogous to the conventional planar-field case, we are led to a formula that consists of scalar invariants of the spectral coherence matrix only. We show that attractive physical insight is gained by expressing the three-dimensional degree of polarization explicitly with the help of the correlations between the three orthogonal spectral components of the electric field. Furthermore, we discuss the fundamental differences in characterizing the polarization state of a field by employing either the two- or the three-dimensional coherence-matrix formalism. The extension of the concept of the degree of polarization to include electromagnetic fields having structures of arbitrary form is expected to be particularly useful, for example, in near-field optics.

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I. INTRODUCTION

The degree of polarization is an important quantity of electromagnetic fields, as it characterizes the correlations that prevail between the orthogonal components of the electric field. Conventionally, the polarization state of a fluctuating electromagnetic field is described in terms of the 2×2 coherence matrix or the related four Stokes parameters [1–3]. The two-dimensional (2D) formalism applies, however, only to fields having planar wave fronts, such as well-collimated and uniform optical beams or radiated wide-angle far fields, which can locally be considered as planar. Such fields can be described by two orthogonal electric field components, but an arbitrary field is generally composed of three components.

In this paper, we focus on the generalization of the concept of the degree of polarization to include also fields with wave fronts of arbitrary form. Such a generalization is useful, for example, for investigations of optical near fields, which are characterized by evanescent waves. The problem at hand has already been studied in the seventies and early eighties, seemingly independently, by Samson and coworkers [4–6] for low-frequency fields relevant to geophysical investigations and by Barakat [7,8] for optical fields. In Ref. [4] Samson approaches the problem by investigating different expansions of the full 3×3 spectral coherence matrix. For one such expansion, he interprets the expansion coefficients as the nine spectral Stokes parameters and defines the degree of polarization in a manner analogous to the two-dimensional case. Much of the same has also been performed by Barakat in Ref. [7]. In congruence, the authors of Refs. [5,6,8] formulate the degree of polarization in terms of scalar invariants, which are traces of different powers of the spectral coherence matrix and its determinant. These invariants appear as coefficients in the characteristic equation of the coherence matrix. Based on such a treatment, Barakat as a matter of fact, proposes in Ref. [8] two measures for the degree of polarization, of which one is the same as that suggested by Samson. More recently, polarization of arbitrary electromagnetic fields has been examined by Brosseau [1,9] in terms of polarization entropy, and by Carozzi et al. [10] in terms of the generalized spectral-density Stokes parameters. As the authors in Refs. [1,6,9] have explicitly noted, the 3×3 coherence matrix cannot, in general, be decomposed into the sum of matrices describing fully polarized and fully unpolarized field as in the two-dimensional case. This fact makes it more difficult to obtain simple physical insight into the proposed formulas for the degree of polarization of arbitrary fields. In this work, we point out some fundamental differences between the two- and three-dimensional coherence-matrix formalisms and give physical intuition into the formulation of the 3D degree of polarization.

We have arranged the paper as follows. In Sec. II, the construction of the degree of polarization for planar wave fields in terms of the 2×2 coherence matrix and the Stokes parameters is recalled to facilitate the subsequent treatment of the degree of polarization of arbitrary wave fields. In Sec. III we formulate the 3D degree of polarization and compare the features of the 2D and 3D formalisms. Finally, in Sec. IV we summarize the main conclusions of the work.

II. DEGREE OF POLARIZATION FOR PLANAR FIELDS

In this section, we examine a planar electromagnetic field propagating in the z direction with the electric field oscillating in the xy plane. We consider a single frequency component $\mathbf{E}(\mathbf{r},\omega)$ of a statistically stationary field and write the corresponding coherence-matrix elements in the space-frequency domain (the spectral-density tensor) as (Sec. 4.7.2 of Ref. [2])

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$$\phi_{ij}(\mathbf{r},\omega)=\langle E_i^*(\mathbf{r},\omega)E_j(\mathbf{r},\omega)\rangle, \qquad i,j=x,y. \tag{1}$$

Here the angle brackets denote averaging, at a point $\mathbf{r}$ over an ensemble of field realizations of frequency ω, and the superscript * stands for complex conjugation. The 2×2 coherence matrix will be denoted by the symbol Φ_2, with the subscript 2 indicating that we are dealing with the two-dimensional formalism. Furthermore, we will implicitly assume $(\mathbf{r},\omega)$ dependency for the coherence matrix, and emphasize that we consider spectral quantities.

The 2×2 coherence matrix is a non-negative definite and Hermitian matrix that entirely specifies the state of polarization of the planar field. It can be uniquely decomposed into a sum of two matrices, one corresponding to fully polarized light and the other to fully unpolarized light. The degree of polarization can then be expressed as the ratio of the intensity (or trace) of the polarized part to the total intensity of the field [1–3]. The resulting expression for the degree of polarization of the two-dimensional field, P_2, has the well-known form

$$P_2^2=1-\frac{4\det(\Phi_2)}{\mathrm{tr}^2(\Phi_2)}=2\left[\frac{\mathrm{tr}(\Phi_2^2)}{\mathrm{tr}^2(\Phi_2)}-\frac{1}{2}\right]. \tag{2}$$

This quantity is bounded to the interval $0\leqslant P_2\leqslant 1$, with the values $P_2=0$ and $P_2=1$ corresponding to a completely unpolarized and polarized field, respectively. It is invariant under unitary transformations, since trace and determinant are scalar invariants under such operations. Due to Hermiticity, the coherence matrix can always be diagonalized by a unitary transformation, and we could readily express the degree of polarization in terms of the eigenvalues of the matrix. The Hermiticity and non-negative definite character of the matrix, respectively, imply that the eigenvalues are real and non-negative.

For this work, particularly relevant is the presentation of the degree of polarization in terms of the four Stokes parameters. The 2D Stokes parameters S_j, $(j=0,\ldots,3)$ are measurable quantities that appear as the expansion coefficients when the coherence matrix is expanded in terms of the 2×2 unit matrix σ_0 and the three Pauli matrices, or generators of the SU(2) symmetry group, σ_j $(j=1,\ldots,3)$, i.e.,

$$\Phi_2=\frac{1}{2}\sum_{j=0}^{3} S_j\sigma_j, \tag{3}$$

where

$$\sigma_0=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},\quad \sigma_1=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},\quad \sigma_2=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},$$

$$\sigma_3=\begin{pmatrix}0 & i\\ -i & 0\end{pmatrix}. \tag{4}$$

This allows us to write the coherence matrix as

$$\Phi_2=\frac{1}{2}\begin{pmatrix}S_0+S_1 & S_2+iS_3\\ S_2-iS_3 & S_0-S_1\end{pmatrix}. \tag{5}$$

Moreover, since $\mathrm{tr}(\sigma_i\sigma_j)=2\delta_{ij}$, one obtains,

$$S_j=\mathrm{tr}(\sigma_j\Phi_2), \qquad \text{or} \qquad \begin{aligned} S_0&=\phi_{xx}+\phi_{yy},\\ S_1&=\phi_{xx}-\phi_{yy},\\ S_2&=\phi_{xy}+\phi_{yx},\\ S_3&=i(\phi_{yx}-\phi_{xy}). \end{aligned} \tag{6}$$

We see that the first Stokes parameter S_0 is proportional to the spectral density of the field. The parameter S_1 describes the excess in spectral density of the x component over that of the y component of the field. The parameter S_2 represents the excess of $+45°$ linearly polarized component over $-45°$ linearly polarized component, and S_3 the excess in spectral density of right-hand circularly polarized field component over left-hand circularly polarized one [3]. Substituting the coherence matrix of Eq. (5) into Eq. (2), the degree of polarization takes the form

$$P_2=\frac{(S_1^2+S_2^2+S_3^2)^{1/2}}{S_0}. \tag{7}$$

When the field is fully polarized, the polarization state can be geometrically represented as a point (S_1,S_2,S_3) on a sphere of radius S_0, the so-called Poincaré sphere. The equator of the sphere (in the S_1S_2 plane) corresponds to linearly polarized light, and the north and south poles to right-hand and left-hand circularly polarized light, respectively. Furthermore, in the origin of the Poincaré sphere the field is fully unpolarized and in every other inner point partially polarized.

Sometimes it is useful to normalize the off-diagonal elements of the coherence matrix by defining

$$\frac{\phi_{xy}}{(\phi_{xx})^{1/2}(\phi_{yy})^{1/2}}\equiv\mu_{xy}=|\mu_{xy}|e^{i\beta_{xy}}, \qquad \mu_{yx}=\mu_{xy}^*. \tag{8}$$

The quantity $|\mu_{xy}|$ is bounded between 0 and 1 and gives a measure for the degree of correlation between the two orthogonal components of the electric field. While the value of the 2D degree of polarization does not depend on the orientation of the orthogonal 2D coordinate system in the plane perpendicular to the wave's propagation direction, the degree of correlation does. One can show that $|\mu_{xy}|\leqslant P_2$, i.e., the maximum value of the degree of correlation is equal to the degree of polarization of the wave. The equality holds in a coordinate system in which the intensities in the x and y directions are equal $(\phi_{xx}=\phi_{yy})$. This situation can always be achieved by a suitable rotation of the coordinate system [3].

III. DEGREE OF POLARIZATION FOR ARBITRARY ELECTROMAGNETIC FIELDS

We now focus on the problem of how the treatment of the planar (two-dimensional) fields could be extended to include arbitrary electromagnetic fields. We proceed analogously to the 2D case, and expand the 3×3 spectral coherence matrix,

$$\phi_{ij}(\mathbf{r},\omega)=\langle E_i^*(\mathbf{r},\omega)E_j(\mathbf{r},\omega)\rangle, \qquad i,j=x,y,z, \tag{9}$$

in the form [1]

$$\Phi_3=\frac{1}{3}\sum_{j=0}^{8}\Lambda_j\lambda_j, \tag{10}$$

where the subscript 3 refers to the 3D formalism. In Eq. (10), λ_0 is the 3×3 unit matrix and the matrices λ_j, ($j=1,\ldots,8$) are the Gell-Mann matrices or the eight generators of the SU(3) symmetry group. The basis matrices are Hermitian, trace orthogonal, and linearly independent. They are explicitly written as [11]

$$\lambda_0=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix},\quad \lambda_1=\begin{pmatrix}0&1&0\\1&0&0\\0&0&0\end{pmatrix},$$

$$\lambda_2=\begin{pmatrix}0&-i&0\\i&0&0\\0&0&0\end{pmatrix},$$

$$\lambda_3=\begin{pmatrix}1&0&0\\0&-1&0\\0&0&0\end{pmatrix},\quad \lambda_4=\begin{pmatrix}0&0&1\\0&0&0\\1&0&0\end{pmatrix},$$

$$\lambda_5=\begin{pmatrix}0&0&-i\\0&0&0\\i&0&0\end{pmatrix}, \tag{11}$$

$$\lambda_6=\begin{pmatrix}0&0&0\\0&0&1\\0&1&0\end{pmatrix},\quad \lambda_7=\begin{pmatrix}0&0&0\\0&0&-i\\0&i&0\end{pmatrix},$$

$$\lambda_8=\frac{1}{\sqrt{3}}\begin{pmatrix}1&0&0\\0&1&0\\0&0&-2\end{pmatrix}.$$

For the basis matrices, the following trace-orthogonality equation holds:

$$\mathrm{tr}(\lambda_i\lambda_j)=\begin{cases}3 & \text{when } i=j=0\\ 2\,\delta_{ij} & \text{otherwise.}\end{cases} \tag{12}$$

On multiplying both sides of Eq. (10) by λ_k, and taking the trace, we may express the expansion coefficients, or the 3D spectral Stokes parameters Λ_k in the form

$$\Lambda_0=\mathrm{tr}(\Phi_3),$$
$$\Lambda_k=\frac{3}{2}\mathrm{tr}(\lambda_k\Phi_3), \quad \text{when} \quad k\geqslant1, \tag{13}$$

or explicitly as,

$$\begin{aligned}
&\Lambda_0=\phi_{xx}+\phi_{yy}+\phi_{zz}, && \Lambda_5=\tfrac{3}{2}i(\phi_{xz}-\phi_{zx}),\\
&\Lambda_1=\frac{3}{2}(\phi_{xy}+\phi_{yx}), && \Lambda_6=\tfrac{3}{2}(\phi_{yz}+\phi_{zy}),\\
&\Lambda_2=\frac{3}{2}i(\phi_{xy}-\phi_{yx}), && \Lambda_7=\tfrac{3}{2}i(\phi_{yz}-\phi_{zy}),\\
&\Lambda_3=\frac{3}{2}(\phi_{xx}-\phi_{yy}), && \Lambda_8=\frac{\sqrt{3}}{2}(\phi_{xx}+\phi_{yy}-2\phi_{zz}).\\
&\Lambda_4=\frac{3}{2}(\phi_{xz}+\phi_{zx}),
\end{aligned} \tag{14}$$

As in the 2D formalism, the first Stokes parameter is proportional to the total spectral density of the field. Moreover, we may interpret the parameters Λ_1 and Λ_2 as playing a role analogous to parameters S_2 and S_3 in the 2D formalism. The same interpretation also holds for the pairs (Λ_4,Λ_5) and (Λ_6,Λ_7), but in the xz and yz planes, respectively. The parameter Λ_3 is obviously analogous to S_1, and Λ_8 represents the sum of the excesses in spectral density in the x and y directions over that in the z-direction. Furthermore, in analogy with the 3D Poincaré sphere, it is possible to characterize the polarization state of a 3D electromagnetic field in terms of a sphere in the eight-dimensional Stokes-parameter space. However, owing to large number of dimensions, such a construction would not provide much geometrical intuition on the subject.

In terms of the Stokes parameters, the 3×3 coherence matrix takes the form

$$\Phi_3=\frac{1}{3}\begin{pmatrix}\Lambda_0+\Lambda_3+\frac{1}{\sqrt{3}}\Lambda_8 & \Lambda_1-i\Lambda_2 & \Lambda_4-i\Lambda_5\\ \Lambda_1+i\Lambda_2 & \Lambda_0-\Lambda_3+\frac{1}{\sqrt{3}}\Lambda_8 & \Lambda_6-i\Lambda_7\\ \Lambda_4+i\Lambda_5 & \Lambda_6+i\Lambda_7 & \Lambda_0-\frac{2}{\sqrt{3}}\Lambda_8\end{pmatrix}. \tag{15}$$

It should be noted that we could have chosen some other complete set of matrices for the basis, and then identified the expansion coefficients as the Stokes parameters. For example, Roman [12] chooses a set of matrices satisfying the Kemmer algebra. However, the choice of the Gell-Mann matrices conveniently leads to the first spectral Stokes parameter being proportional to the total spectral density of the field, as well as to the other parameters having physical meanings analogous to those of the 2D Stokes parameters. As in the 2D formalism, we can also define the degree of correlation $|\mu_{ij}|$ $(0\leqslant|\mu_{ij}|\leqslant 1)$ between any two of the three orthogonal electric field components as

$$\frac{\phi_{ij}}{(\phi_{ii})^{1/2}(\phi_{jj})^{1/2}}\equiv\mu_{ij}=|\mu_{ij}|e^{i\beta_{ij}},$$

$$\mu_{ji}=\mu_{ij}^*, \quad i,j=x,y,z. \tag{16}$$

Owing to the fact that the 3×3 coherence matrix cannot, in general, be decomposed into the sum of a fully polarized and fully unpolarized part [1,6,9], other definitions for the degree of polarization of 3D fields must be sought for. Let us now investigate the possibility of expressing the 3D degree of polarization P_3 in the form

$$P_3^2=\frac{1}{3}\,\frac{\sum_{j=1}^{8}\Lambda_j^2}{\Lambda_0^2}. \tag{17}$$

This form is analogous to Eq. (7), and it has previously been put forward by Samson [4] and Barakat [7], although, in those works a different coefficient appears in front of the expression owing to the slightly different basis matrices. On substituting the Stokes parameters of Eq. (14) into Eq. (17), the 3D degree of polarization can be expressed in terms of the coherence matrix Φ_3 as

$$P_3^2=\frac{3}{2}\left[\frac{\mathrm{tr}(\Phi_3^2)}{\mathrm{tr}^2(\Phi_3)}-\frac{1}{3}\right]. \tag{18}$$

We see that Eq. (18) is invariant under unitary transformations, and consequently, the value of the degree of polarization is independent of the orientation of the orthogonal coordinate system. Furthermore, due to the Hermiticity, we may diagonalize the coherence matrix and write

$$\frac{\mathrm{tr}(\Phi_3^2)}{\mathrm{tr}^2(\Phi_3)}=\frac{a_1^2+a_2^2+a_3^2}{(a_1+a_2+a_3)^2}, \tag{19}$$

where (a_1,a_2,a_3) are the eigenvalues of the coherence matrix. On expanding the denominator, and noting that all the eigenvalues are non-negative, we see that $\mathrm{tr}(\Phi_3^2)/\mathrm{tr}^2(\Phi_3)\leqslant 1$. Moreover, by applying the Cauchy-Schwarz inequality we see that $\mathrm{tr}(\Phi_3^2)/\mathrm{tr}^2(\Phi_3)\geqslant 1/3$. It then follows that

$$0\leqslant P_3\leqslant 1, \tag{20}$$

as is required for a measure of the degree of polarization.

We next reduce the formula for the 3D degree of polarization to correspond to the case of planar fields. For example, by setting $E_z=0$, we consider a field that oscillates in the xy plane. We thus obtain $\phi_{xz}=\phi_{zx}=\phi_{yz}=\phi_{zy}=\phi_{zz}=0$, and consequently from Eq. (14) that $\Lambda_4=\Lambda_5=\Lambda_6=\Lambda_7=0$ and $\Lambda_8=\sqrt{3}/2\Lambda_0$. The coherence matrix Φ_3 then reduces to

$$\Phi_3=\frac{1}{3}\begin{pmatrix}\frac{3}{2}\Lambda_0+\Lambda_3 & \Lambda_1-i\Lambda_2 & 0\\ \Lambda_1+i\Lambda_2 & \frac{3}{2}\Lambda_0-\Lambda_3 & 0\\ 0 & 0 & 0\end{pmatrix}. \tag{21}$$

Comparing the expressions for Λ_j, $(j=0,\ldots,3)$ with the Stokes parameters of the 2D fields, Eq. (6), we find that the 2×2 matrix in the upper left corner of Φ_3 is exactly the same as the matrix of Eq. (5). Let us denote that matrix by Φ_2'. We may now rewrite Eq. (18) for a field characterized by the coherence matrix of Eq. (21) as

$$P_{3\to 2}^2=1-\frac{3\,\det(\Phi_2')}{\mathrm{tr}^2(\Phi_2')}. \tag{22}$$

Now a fundamental difference between the 2D and 3D formalisms emerges. The values for the degree of polarization of a 2D field calculated in terms of the 2D and 3D formalisms are not, in general, equal which is indicated by the factor 3 in Eq. (22) instead of the factor 4 that is present in Eq. (2). Writing the factor $3\det(\Phi_2')/\mathrm{tr}^2(\Phi_2')$ in terms of the eigenvalues of Φ_2', which are non-negative, and noting that their geometric mean value is smaller than or equal to the arithmetic mean value, we find that

$$\frac{1}{2}\leqslant P_{3\to 2}\leqslant 1. \tag{23}$$

Thus, a planar field cannot be fully unpolarized in the 3D formalism. This is as expected, since in such a field the oscillations are restricted to a single plane, and consequently, when treated as three-dimensional the field must have a nonzero degree of polarization. Since the degree of polarization retains its value under a rotation of the coordinate system, Eq. (23) is valid for any 2D field.

The most intuitive understanding of the differences between the 2D and 3D formalisms is, perhaps, obtained by considering Fig. 1. In the upper row an unpolarized 2D field, i.e., a field for which the spectral density in the x and y directions is the same $(\phi_{xx}=\phi_{yy})$, and for which no correlation exists between the two electric field components $(|\mu_{xy}|=0)$, is passed through a polarizer. The 2D formalism gives the values $P_2=0$ and $P_2=1$ for the field before and after the polarizer, respectively. Let us now consider 3D fields in a similar way. Assume a fully unpolarized 3D field (lower row in Fig. 1), which is polarized by two devices each cutting off one of the orthogonal field components. For a fully unpolarized 3D field the spectral density in all three orthogonal directions is the same $(\phi_{11}=\phi_{22}=\phi_{33})$ and no

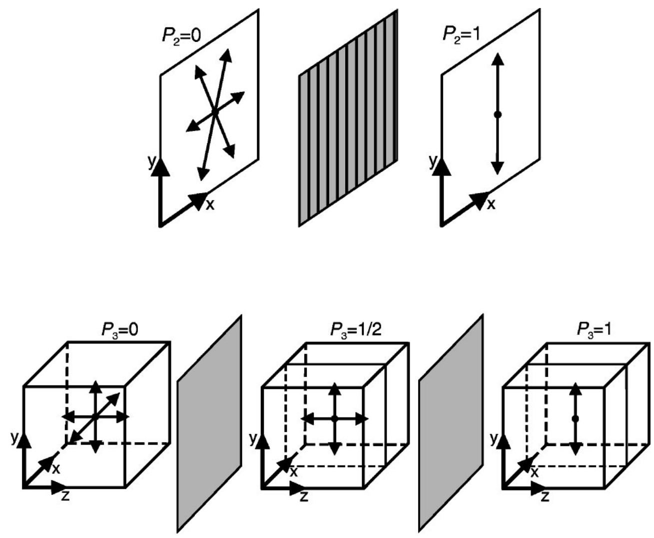

FIG. 1. A geometric illustration of the differences between the 2D and 3D coherence-matrix formalisms in treating the polarization state of an electromagnetic field.

correlations exist between any of the electric field components ($|\mu_{xy}|=|\mu_{xz}|=|\mu_{yz}|=0$). For this field, which cannot be described in terms of the 2D formalism, the 3D formalism gives the value of $P_3=0$. When the x component of the field is cut off by the first device the field becomes partially polarized. Indeed, now $\phi_{xx}=0$, $\phi_{yy}=\phi_{zz}$ with $|\mu_{yz}|=0$, and consequently $P_3=1/2$. The second device then cuts off the z component and the field becomes fully polarized ($P_3=1$), since the oscillations now take place only in a single direction. We may conclude that the fundamental difference between the 2D and 3D formalism is due to the fact that in the latter, the third direction is included albeit the intensity in this direction may be zero. In the 2D formalism the third direction is not even considered, and therefore, that formalism cannot be applied to characterize polarization of an arbitrary field.

Let us consider the 3D counterpart of the statement in 2D, $P_2 \geqslant |\mu_{xy}|$ [see the discussion below Eq. (8)]. Samson has investigated the subject by extending the analysis from the real coordinate space to the complex unitary space [6]. He showed that in the unitary space, the maximum value of the degree of correlation between the field components in two orthogonal (complex) directions is, unlike in the 2D case, greater than the 3D degree of polarization. However, since the analysis is performed in the unitary space, the result lacks a direct physical explanation. Here we perform the analysis in the real coordinate space, which allows a physically intuitive connection to be made between the field correlations and the 3D degree of polarization. We proceed by applying Eq. (16), and rewrite Eq. (18) in the form

$$1-P_3^2=3\frac{(1-|\mu_{xy}|^2)\phi_{xx}\phi_{yy}+(1-|\mu_{xz}|^2)\phi_{xx}\phi_{zz}+(1-|\mu_{yz}|^2)\phi_{yy}\phi_{zz}}{(\phi_{xx}+\phi_{yy}+\phi_{zz})^2}, \tag{24}$$

or as

$$1-P_3^2=3\left(1-\frac{|\mu_{xy}|^2\phi_{xx}\phi_{yy}+|\mu_{xz}|^2\phi_{xx}\phi_{zz}+|\mu_{yz}|^2\phi_{yy}\phi_{zz}}{\phi_{xx}\phi_{yy}+\phi_{xx}\phi_{zz}+\phi_{yy}\phi_{zz}}\right)\Bigg/\left(\frac{\phi_{xx}^2+\phi_{yy}^2+\phi_{zz}^2}{\phi_{xx}\phi_{yy}+\phi_{xx}\phi_{zz}+\phi_{yy}\phi_{zz}}+2\right). \tag{25}$$

Then, by noting that for any set of three real numbers (a,b,c)

$$(a-b)^2+(a-c)^2+(b-c)^2\geqslant 0$$
$$\Leftrightarrow a^2+b^2+c^2\geqslant ab+ac+bc, \tag{26}$$

we find that

$$P_3^2\geqslant\frac{|\mu_{xy}|^2\phi_{xx}\phi_{yy}+|\mu_{xz}|^2\phi_{xx}\phi_{zz}+|\mu_{yz}|^2\phi_{yy}\phi_{zz}}{\phi_{xx}\phi_{yy}+\phi_{xx}\phi_{zz}+\phi_{yy}\phi_{zz}}. \tag{27}$$

Equation (27) has a simple physical interpretation. It states that the square of the 3D degree of polarization represents the upper limit of the average of the squared correlations weighted by the corresponding spectral densities. In fact, this is intuitively reasonable, since the degree of polarization is determined by the correlations between the three orthogonal electric field components and their intensities. The value of the right-hand side of Eq. (27) depends on the orientation of the coordinate system, but the left-hand side does not. The right-hand side reaches the value of P_3^2 if the coordinate system is oriented in such a way that $\phi_{xx}=\phi_{yy}=\phi_{zz}$. In this case, the equality sign holds, and we obtain

$$P_3^2=\frac{|\mu_{xy}|^2+|\mu_{xz}|^2+|\mu_{yz}|^2}{3}, \tag{28}$$

indicating that the square of the 3D degree of polarization is equal to the pure average of the squared correlations prevailing between the three orthogonal electric field components in this specific coordinate system. This result agrees well with intuitive physical meaning of the degree of polarization.

On the other hand, in the special case when the intensity in one direction is zero, say in the z direction, Eq. (24) reduces to

$$1-P_3^2=3(1-|\mu_{xy}|^2)\frac{\phi_{xx}\phi_{yy}}{(\phi_{xx}+\phi_{yy})^2}. \tag{29}$$

Since $\phi_{xx}\phi_{yy}/(\phi_{xx}+\phi_{yy})^2\leqslant 1/4$, we obtain

$$P_3^2\geqslant\frac{1}{4}+\frac{3}{4}|\mu_{xy}|^2. \tag{30}$$

This is consistent with Eq. (23), which states that the 3D degree of polarization of a planar field cannot be lower than $P_3=1/2$. As previously, the equality holds when $\phi_{xx}=\phi_{yy}$, and we see that for a planar field the 3D degree of polarization is directly related to the correlation that exists between the two nonzero electric field components.

We have enclosed in the Appendix a proof that there always exist three mutually orthogonal directions for which the spectral intensities are equal. In this system, the square of the degree of polarization is equal to the pure average of the squared correlations, as stated by Eq. (28). Based on these arguments, we propose that Eq. (27) together with Eq. (28) justifies Eq. (18), or alternatively Eq. (17), to be considered a sensible measure for the 3D degree of polarization, as they relate the degree of polarization to the correlations that exist between the three electric field components of an arbitrary field.

IV. CONCLUSION

We have formulated an extension to the concept of degree of polarization that is applicable for arbitrary electromagnetic fields. Our formula for the 3D degree of polarization is consistent with the results that have been put forward in the literature already some time ago. However, our way of formulating the concept in a manner that is analogous to that of the well-established 2D coherence-matrix formalism brings along a different physical insight into the subject matter. We demonstrated that the dimensionality (2D vs 3D) is a crucial issue for the quantitative value and interpretation of the results. We also showed how the 3D degree of polarization may be interpreted as a quantity that characterizes the correlations between all three orthogonal electric field components. The presented form for the 3D degree of polarization is expected to be a useful tool in assessing the partial polarization of non-planar electromagnetic fields such as optical near fields. Reference [13] provides an example of the use of this formalism to analyze the degree of polarization of thermal near fields under the influence of resonant surface waves.

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APPENDIX

We show that for every coherence matrix, we can rotate the coordinate system in such a way that the diagonal elements become equal. Let us perform two successive rotations of which the first is chosen to be about the z-axis counterclockwise through an angle α, and the second about the y'-axis counterclockwise through an angle β. The corresponding rotation matrices and the elements of the coherence matrix after each rotation are listed as follows:

$$\begin{pmatrix} E_x' \\ E_y' \\ E_z' \end{pmatrix}=\begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}, \tag{A1}$$

$$\begin{aligned}
\phi_{xx}'&=\cos^2\alpha\,\phi_{xx}+\sin^2\alpha\,\phi_{yy}+1/2\sin 2\alpha(\phi_{xy}+\phi_{yx}),\\
\phi_{yy}'&=\sin^2\alpha\,\phi_{xx}+\cos^2\alpha\,\phi_{yy}-1/2\sin 2\alpha(\phi_{xy}+\phi_{yx}),\\
\phi_{zz}'&=\phi_{zz},\\
\phi_{xy}'&=1/2\sin 2\alpha(\phi_{yy}-\phi_{xx})+\cos^2\alpha\,\phi_{xy}-\sin^2\alpha\,\phi_{yx},\\
\phi_{xz}'&=\cos\alpha\,\phi_{xz}+\sin\alpha\,\phi_{yz},\\
\phi_{yz}'&=-\sin\alpha\,\phi_{xz}+\cos\alpha\,\phi_{yz},\\
\phi_{ij}'&=\phi_{ji}'^{*}, \qquad i\neq j,
\end{aligned} \tag{A2}$$

$$\begin{pmatrix} E_x'' \\ E_y'' \\ E_z'' \end{pmatrix}=\begin{pmatrix} \cos\beta & 0 & -\sin\beta \\ 0 & 1 & 0 \\ \sin\beta & 0 & \cos\beta \end{pmatrix}\begin{pmatrix} E_x' \\ E_y' \\ E_z' \end{pmatrix}, \tag{A3}$$

$$\begin{aligned}
\phi_{xx}''&=\cos^2\beta\,\phi_{xx}'+\sin^2\beta\,\phi_{zz}'-1/2\sin 2\beta(\phi_{xz}'+\phi_{zx}'),\\
\phi_{yy}''&=\phi_{yy}',\\
\phi_{zz}''&=\sin^2\beta\,\phi_{xx}'+\cos^2\beta\,\phi_{zz}'+1/2\sin 2\beta(\phi_{xz}'+\phi_{zx}'),\\
\phi_{xy}''&=\cos\beta\,\phi_{xy}'-\sin\beta\,\phi_{zy}',\\
\phi_{xz}''&=1/2\sin 2\beta(\phi_{xx}'-\phi_{zz}')+\cos^2\beta\,\phi_{xz}'-\sin^2\beta\,\phi_{zx}',\\
\phi_{yz}''&=\sin\beta\,\phi_{yx}'+\cos\beta\,\phi_{yz}',\\
\phi_{ij}''&=\phi_{ji}''^{*}, \qquad i\neq j.
\end{aligned} \tag{A4}$$

We proceed by requiring that in the final coordinate system $\phi_{xx}''=\phi_{yy}''=\phi_{zz}''$. On applying Eq. (A4), the condition $\phi_{xx}''=\phi_{zz}''$ gives

$$\tan 2\beta=\frac{\phi_{xx}'-\phi_{zz}'}{\phi_{xz}'+\phi_{zx}'}. \tag{A5}$$

If the angle β is as in Eq. (A5), we obtain from Eq. (A4) that $\phi_{xx}''=\phi_{zz}''=(\phi_{xx}'+\phi_{zz}')/2$. Furthermore, since the element ϕ_{yy}' does not change under the β rotation, we have

$$\phi_{yy}'=\frac{\phi_{xx}'+\phi_{zz}'}{2}, \tag{A6}$$

TABLE I. Rotations that lead to equal diagonal elements for the coherence matrix ($\phi_{xx}=\phi_{yy}=\phi_{zz}$), and the sufficient conditions for the rotation angles to be real. The chosen rotations are determined solely by the relative values of the diagonal elements. Every coherence matrix belongs at least to one of these categories.

	Rotations (angle, axis)	Conditions for the angles α and β to be real
1	(α,z) and (β,y')	$\phi_{xx}\leqslant \mathrm{tr}(\Phi_3)/3$ and $\phi_{yy}\geqslant \mathrm{tr}(\Phi_3)/3$, $\phi_{xx}\neq\phi_{yy}$
2	(α,z) and (β,y')	$\phi_{xx}\geqslant \mathrm{tr}(\Phi_3)/3$ and $\phi_{yy}\leqslant \mathrm{tr}(\Phi_3)/3$, $\phi_{xx}\neq\phi_{yy}$
3	(α,x) and (β,z')	$\phi_{yy}\leqslant \mathrm{tr}(\Phi_3)/3$ and $\phi_{zz}\geqslant \mathrm{tr}(\Phi_3)/3$, $\phi_{yy}\neq\phi_{zz}$
4	(α,x) and (β,z')	$\phi_{yy}\geqslant \mathrm{tr}(\Phi_3)/3$ and $\phi_{zz}\leqslant \mathrm{tr}(\Phi_3)/3$, $\phi_{yy}\neq\phi_{zz}$

which fixes the angle α. By substituting the primed diagonal elements from Eq. (A2) into Eq. (A6) we are led to the condition

$$\cos 2\alpha(\phi_{yy}-\phi_{xx})-\sin 2\alpha(\phi_{xy}+\phi_{yx})=\frac{2\phi_{zz}-\phi_{xx}-\phi_{yy}}{3}, \tag{A7}$$

for α. Equivalently, Eq. (A7) may be expressed in the form

$$\sin(2\alpha+\varphi)=\frac{2\phi_{zz}-\phi_{xx}-\phi_{yy}}{3\sqrt{(\phi_{xy}+\phi_{yx})^2+(\phi_{yy}-\phi_{xx})^2}}, \tag{A8}$$

where

$$\tan\varphi=\frac{\phi_{xx}-\phi_{yy}}{\phi_{xy}+\phi_{yx}}, \tag{A9}$$

and where the quadrant of φ is chosen such that $-\phi_{xy}-\phi_{yx}$ and $\cos\varphi$, as well as $\phi_{yy}-\phi_{xx}$ and $\sin\varphi$, have the same sign. It is of interest to note that both angles α and β can be expressed solely in terms of the Stokes parameters of Eq. (14).

Since the coherence matrix is Hermitian, the angle φ is always real as is seen from Eq. (A9). Therefore, the condition for α to be real, which according to Eqs. (A5) and (A2) implies that also β is real, is that the right-hand side of Eq. (A8) is bounded between -1 and 1. This is true *at least* when

$$-1\leqslant\frac{2\phi_{zz}-\phi_{xx}-\phi_{yy}}{3|\phi_{yy}-\phi_{xx}|}\leqslant 1. \tag{A10}$$

This equation is satisfied, if $\phi_{xx}\leqslant \mathrm{tr}(\Phi_3)/3$, and $\phi_{yy}\geqslant \mathrm{tr}(\Phi_3)/3$ or if $\phi_{xx}\geqslant \mathrm{tr}(\Phi_3)/3$ and $\phi_{yy}\leqslant \mathrm{tr}(\Phi_3)/3$. For both cases we also require that ϕ_{yy} and ϕ_{xx} are not both equal to $\mathrm{tr}(\Phi_3)/3$. In other words, if the spectral intensity of the field in the x direction is smaller than or equal to, and in the y direction greater than or equal to one third of the total spectral intensity, or vice versa, the angles α and β are both real.

When the above conditions for the angles α and β are not met, we choose a different pair of rotation axes. For example, we first rotate about the x-axis counterclockwise through an angle α, and then about the z'-axis counterclockwise through an angle β. The angles associated with these rotations are obtained simply by performing a cyclic permutation for the labels of the coordinate axes, i.e., we replace x with y, y with z, and z with x, in Eqs. (A5), (A8), and (A9). We then have that the angles α and β for this pair of rotations are real *at least* when $\phi_{yy}\leqslant \mathrm{tr}(\Phi_3)/3$ and $\phi_{zz}\geqslant \mathrm{tr}(\Phi_3)/3$, or when $\phi_{yy}\geqslant \mathrm{tr}(\Phi_3)/3$ and $\phi_{zz}\leqslant \mathrm{tr}(\Phi_3)/3$. Again, both ϕ_{yy} and ϕ_{zz} cannot be equal to $\mathrm{tr}(\Phi_3)/3$ at the same time. In Table I, we have summarized the rotations and the corresponding conditions for the rotation angles to be real. We see that the diagonal elements of every coherence matrix, except the one with $\phi_{xx}=\phi_{yy}=\phi_{zz}$ for which no rotations are needed, fulfill the conditions at least in one category. For example, if $\phi_{xx}>\mathrm{tr}(\Phi_3)/3$ and $\phi_{zz}>\mathrm{tr}(\Phi_3)/3$, then necessarily $\phi_{yy}<\mathrm{tr}(\Phi_3)/3$, and we may apply either rotation 2 or 3 of Table I.

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